Special Factorizations

Kerry Tarrant

May 15, 2022

This review of special factorizations contains a summary of concepts in factorization for functions and in simplifying rational functions.

1 Factorization

Factoring a number is the process of decomposing a number into a product. Below is an example of decomposing the number 12 into a product of prime numbers.

Example 1.1
First note $12 = 3 \cdot 4$.
Notice that 4 can be decomposed into a product of prime numbers: $4 = 2 \cdot 2 = 2^2$.
Thus, $12 = 3 \cdot 4 = 3 \cdot 2 \cdot 2 = 3 \cdot 2^2$.
Here, 12 has been factored into a product of prime numbers.

■

The same idea can be applied to functions. Namely, if you are given a function (like a polynomial), you can decompose the function into a product of its factors. Let's look at one example of this process with the quadratic function $p(x) = 2x^2 + 4x + 2$.

Example 1.2
Notice that the number 2 is a common factor in all the terms of this expression. Factoring out the number 2 then gives us $p(x) = 2x^2 + 4x + 2 = 2(x^2 + 2x + 1)$.
Looking at the expression inside the parentheses, you will notice that the coefficient in front of the second-degree term is just 1. Recall the trick for factorizing expressions that look like the one in the parentheses. We need two numbers that satisfy two conditions: i) when the two numbers are summed together we get the first-degree coefficient 2, and ii) when the two numbers are multiplied together we get the y-intercept coefficient 1.
It's easy to see that the two numbers that satisfy these two conditions are the numbers 1 and 1.
Now $(x^2 + 2x + 1)$ can be factored: $(x^2 + 2x + 1) = (x + 1)(x + 1) = (x + 1)^2$.
And finally we can factor the original expression:
$p(x) = 2x^2 + 4x + 2 = 2(x + 1)^2$.

■

Factorizing numbers and functions can be helpful when attempting to simplifying. Let's start with a small example: the rational number $\frac{12}{4}$ has a numerator and denominator that we factored already in Example 1. Rewriting $\frac{12}{4} = \frac{3 \cdot 2^2}{2^2}$ we can see that the 2^2 cancel. This gives us $\frac{12}{4} = \frac{3 \cdot \cancel{2^2}}{\cancel{2^2}} = \frac{3}{1} = 3$

Factorizing rational functions can also simplify your expression. You will need to be careful of the domain though. If there was a value that made your expression divide by zero, you will still have to exclude that value from your domain. Lets consider an example with the expression $q(x) = \frac{2x^2+4x+2}{x+1}$.

Example 1.3
Notice that we already factorized the numerator in Example 2.
Rewriting gives $q(x) = \frac{2x^2+4x+2}{x+1} = \frac{2(x+1)^2}{x+1}$. Notice again that the numerator and denominator have a common factor $x+1$ that cancels.
Cancelling the common factor leads to $q(x) = \frac{2(x+1)\cancel{(x+1)}}{\cancel{x+1}} = 2(x+1)$.
As mentioned, the domain is still restricted. In the original expression $x \neq -1$ otherwise we would be dividing by zero. All other values for x are okay. In interval notation, the domain of $q(x) = 2(x+1)$ is $(-\infty, -1) \cup (-1, \infty)$.

■

In general, if you were given a rational function $\frac{f(x)}{g(x)}$ where the denominator is factored into $g(x) = g_1(x)g_2(x)$ and the numerator is factored into $f(x) = f_1(x)f_2(x)g_1(x)g_2(x)$ you can cancel common factors between the numerator and denominator such that

$$\frac{f(x)}{g(x)} = \frac{f_1(x)f_2(x)g_1(x)g_2(x)}{g_1(x)g_2(x)} = \frac{f_1(x)f_2(x)\cancel{g_1(x)}\cancel{g_2(x)}}{\cancel{g_1(x)}\cancel{g_2(x)}} = f_1(x)f_2(x).$$

If you were given polynomial of degree n called $P_n(x)$, factorizing the polynomial can be helpful for finding the solutions $x_1, x_2, \ldots, x_n$ such that $P(x) = 0$ (i.e., where the polynomial intersects the x-axis). Factorizing can also help define the domain of your rational function.

2 Special Factorizations

Occasionally, we may come across expressions that might be a little challenging to evaluate. Below are four expressions and their factorized form.

1) $x^2 - y^2 = (x+y)(x-y)$
2) $x^2 + 2xy + y^2 = (x+y)^2$
3) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
4) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

Let's look at an example of each.

Example 2.1

$$x^2 - 144$$

$$= x^2 - 12^2$$

$$= (x+12)(x-12)$$

■

Example 2.2

$$16 + 8x + x^2$$

$$= 4^2 + 2 \cdot 4x + x^2$$

$$= (4+x)^2$$

■

Example 2.2

$$25 - \frac{1}{5}y^3$$

$$= \frac{1}{5}(125 - y^3)$$

$$= \frac{1}{5}(5^3 - y^3)$$

$$= \frac{1}{5}(5 - y)(5^2 + 5y + y^2)$$

■

Example 2.2

$$8x^3 + 64$$

$$= 8(x^3 + 8)$$

$$= 8(x^3 + 2^3)$$

$$= 8(x + 2)(x^2 - 2x + 2^2)$$

■