Vectors for problems 1 & 2:

$A=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $B=\begin{pmatrix}-3\\6\\-3\end{pmatrix}$, $C=\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $D=\begin{pmatrix}-2\\4\\-2\end{pmatrix}$

Form D

$B=\frac{3}{2}D$

$\begin{pmatrix}-3\\6\\-3\end{pmatrix}=\frac{3}{2}\begin{pmatrix}-2\\4\\-2\end{pmatrix}=\begin{pmatrix}-3\\6\\-3\end{pmatrix}$ ✓

1 Which vectors are parallel?

a) A & B b) A & C c) B & D d) B & C e) C & D

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2 Which vectors are perpendicular?

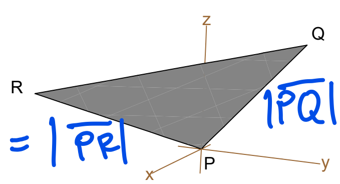
$A\cdot C=1-2+1=0$

a) A & B b) A & C c) A & D d) B & C e) C & D

Problems 3 & 4 use the triangle with vertices at:

$P=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $Q=\begin{pmatrix}1\\2\\2\end{pmatrix}$, $R=\begin{pmatrix}2\\-2\\1\end{pmatrix}$

$|R-Q|=\left|\begin{pmatrix}1\\-4\\-1\end{pmatrix}\right|=\sqrt{18}=3\sqrt{2}$



$3=|PR|$

$|PQ|=3$

$3\sqrt{2}>3$

Two sides are the same length

$\Rightarrow$ Isosceles triangle

3 What is the length of the longest side of the triangle PQR?

a) 1 b) 2 c) 3 d) $3\sqrt{2}$ e) $\sqrt{5}$

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4 What are the angles of the triangle P-Q-R?

a) 50°, 50°, 80° ~~b)~~ 30°, 60°, 90° ~~c)~~ 60°, 60°, 60°

d) 45°, 45°, 90° e) 55°, 55°, 70°

$\angle RPQ=\frac{R\cdot Q}{|R||Q|}=\frac{2-4+2}{3\cdot 3}$

$=0$

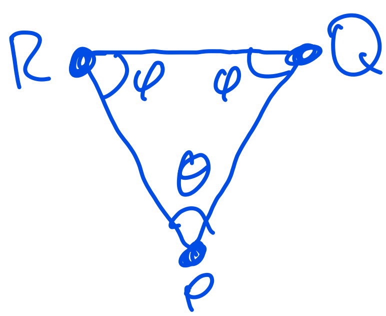
$=\cos\theta$

$\Rightarrow\theta=\frac{\pi}{2}$

i.e. 90°

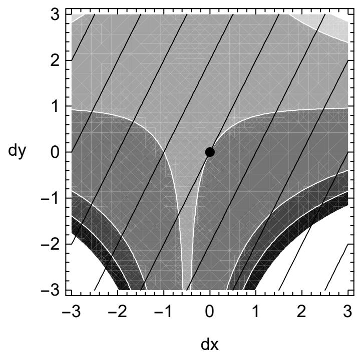
$\varphi=\frac{180°-\theta}{2}$

$=\frac{180°-90°}{2}=45°$



5 The gradient of $f[x, y] = x^2 y + x^2 y^3$ at the point $(x, y) = (1/2, -1)$ is:

a) $G(-2, 4)$ b) $G(-2, 3)$ c) $G(-2, 1)$ d) $G(2, 3)$ e) $G(4, -2)$



$$\nabla f = \begin{pmatrix} 2xy + 2xy^3 \\ x^2 + 3x^2y^2 \end{pmatrix}$$

$$\nabla f\left(\tfrac{1}{2}, -1\right) = \begin{pmatrix} -1 + (-1) \\ \frac{1}{4} + 3\frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

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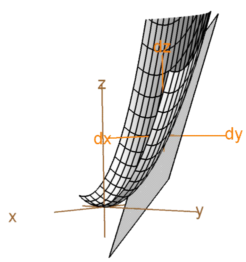
6 The equation of the plane tangent to $z = \frac{x^2}{3} + \frac{y^2}{4}$ at $(x, y, z) = (-3, 2, 4)$ is:

a) $(z - 4) = -(x + 2) + 2(y - 1)$ b) $(z - 4) = (x + 2) + 2(y - 3)$ c) $(z - 4) = -(x - 2) - 2(y - 3)$

d) $(z - 4) = (x + 2) - 2(y + 3)$ e) $(z - 4) = -2(x + 3) + (y - 2)$



$$dz = \frac{2}{3}x\,dx + \frac{2}{4}y\,dy$$

$$\Rightarrow dz = -2dx + dy$$

$$\Rightarrow (z-4) = -2(x+3) + (y-2)$$

7 Parametric equations for the line tangent to the parametric curve $x = \sqrt{2t}$, $y = t^3/2$, $z = 2/t$ **at the point where** $t = 2$ **are:**

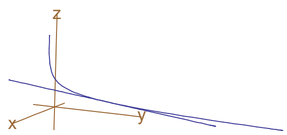
a) $x = 4 - 2s$, $y = 1 + s$, $z = 3 - s/2$

b) $x = 1/2 + 2s$, $y = 3 + 4s$, $z = s - 1$

c) $x = 2 + s/2$, $y = 4 + 6s$, $z = 1 - s/2$

d) $x = 1/(\sqrt{2s})$, $y = 3s^2/2$, $z = -2/s^2$

e) $x - 2 = 1/(\sqrt{2s})$, $y - 4 = 3s^2/2$, $z - 1 = -2/s^2$



$$P(2) = \begin{pmatrix} \sqrt{2\cdot 2} \\ 2^3/2 \\ 2/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$P'(t) = \begin{pmatrix} \frac{1}{2}(2t)^{-1/2}\cdot 2 \\ \frac{3}{2}t^2 \\ -2t^{-2} \end{pmatrix} = \begin{pmatrix} (2t)^{-1/2} \\ \frac{3}{2}t^2 \\ -2t^{-2} \end{pmatrix}$$

$$P'(2) = \begin{pmatrix} \frac{1}{2} \\ 6 \\ -\frac{1}{2} \end{pmatrix}$$

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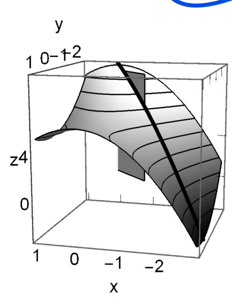
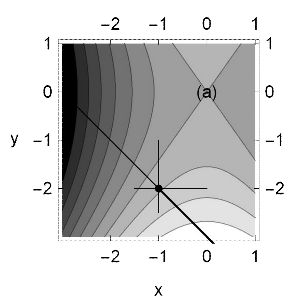
8 The fastest instantaneous rate of increase of the function $f[x, y] = 6 - x^2 + \frac{y^2}{2}$ **at the point** $(x, y) = (-1, -2)$ **is:**

a) $\sqrt{2}$ b) $2\sqrt{2}$ c) $\sqrt{5}$ d) $2/\sqrt{5}$ e) 3





$$\nabla f = \begin{pmatrix} -2x \\ y \end{pmatrix}$$

$$\nabla f(-1,-2) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$|\nabla f(-1,-2)| = \sqrt{8} = 2\sqrt{2}$$

9 The total differential of $z = f[x, y]$ when $f[x, y] = (x^2 - x)(1 + y^2)$

is: $dz =$

a) $(2x - 1) + 2y$

b) $(2x - 1)(1 + y^2)\,dx + 2y(x^2 - x)\,dy$

c) $(2x - 1)\,dx + 2y\,dy$

d) $2(x - 1)xy + (2x - 1)(1 + y^2)$

e) $-1 + 2x - 2xy + 2x^2 y - y^2 + 2xy^2$

$\nabla z = (1+y^2)(2x-1)dx + (x^2-x)(2y)dy$

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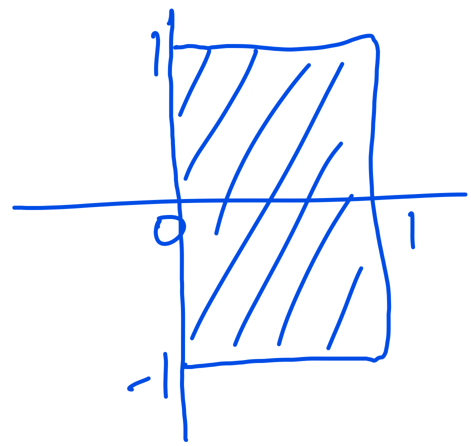
10 When $f[x, y] = x\,y^2$ and $\mathbb{D} = \{(x, y) : 0 \le x \le 1, -1 \le y \le 1\}$,

the area-weighted integral $\iint_{\mathbb{D}} f[x, y]\, da =$

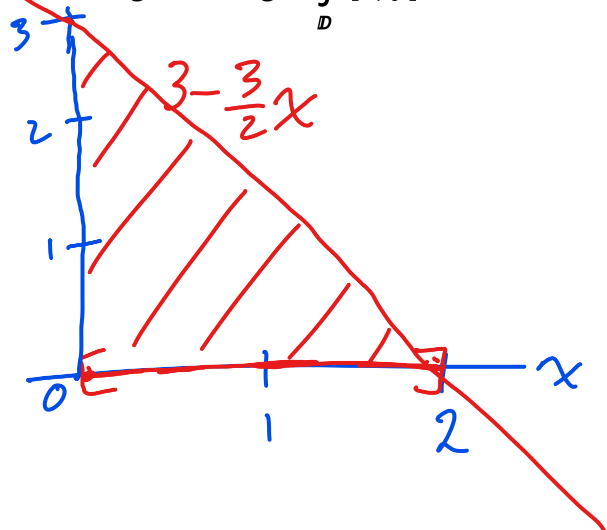
a) 0 b) 1/6 c) 1/3 d) 2/3 e) 4/3



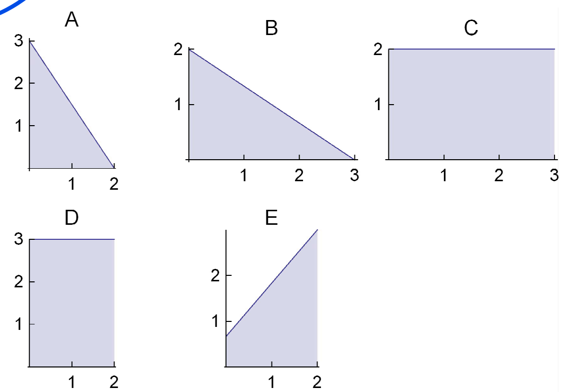
$$\int_0^1 \int_{-1}^1 xy^2\,dy\,dx = \int_0^1 x \int_{-1}^1 y^2\,dy\,dx$$

$$= \int_0^1 x\,\frac{1}{3}\left(1^3 - (-1)^3\right)dx = \frac{2}{3}\int_0^1 x\,dx = \frac{2}{3}\cdot\frac{1}{2}(1^2 - 0) = \frac{1}{3}$$

11 The iterated integral $\int_0^2\left(\int_0^{3-\frac{3}{2}x} f[x,y]\,dy\right)dx$ equals the 2D area-weighted integral $\int_{\mathbb{D}} f[x,y]\,da$ for which of the domains:



a) *A* b) *B* c) C d) *D* e) E



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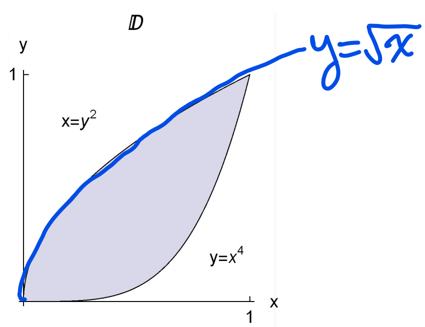
12 For $f[x,y] = 3x^2 y$ over the region bounded by $y = x^4$ and $x = y^2$, the area-weighted integral $\iint_{\mathbb{D}} f[x,y]\,da =$

a) −6/77 b) −3/14 c) 1/2 d) 21/88 e) 1/4



$$\int_0^1\int_{x^4}^{\sqrt{x}} 3x^2y\,dy\,dx = 3\int_0^1 x^2\frac{1}{2}(x-x^8)\,dx$$

$$=\frac{3}{2}\int_0^1(x^3-x^{10})\,dx$$

$$=\frac{3}{2}\left(\frac{1}{4}x^4-\frac{1}{11}x^{11}\right)\Big|_0^1$$

$$=\frac{3}{2}\left(\frac{1}{4}-\frac{1}{11}\right)$$

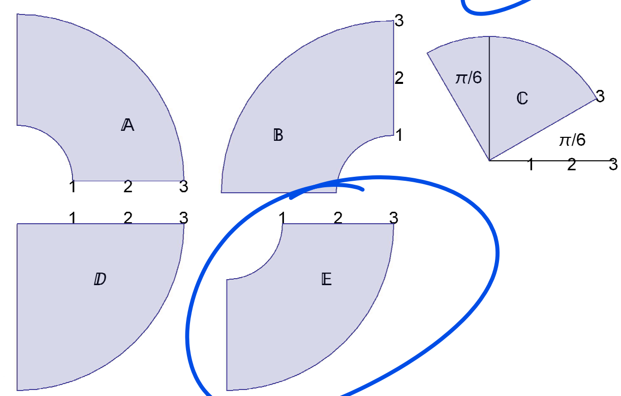
$$=\frac{3}{2}\left(\frac{11-4}{44}\right)$$

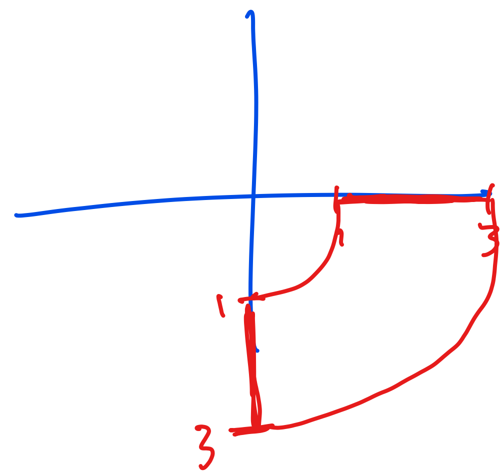
$$=\frac{3}{2}\left(\frac{7}{44}\right)$$

$$=\frac{21}{88}$$

13 The iterated integral $\int_1^3\left(\int_{-\pi/2}^{0} f[r\,\mathrm{Cos}[\theta],\ r\,\mathrm{Sin}[\theta]]\ r\,d\theta\right)dr$ equals the 2D area-weighted integral $\int_{\mathbb{D}} f[x,\ y]\,da$ for which of the domains:

a) $\mathbb{A}$ b) $\mathbb{B}$ c) $\mathbb{C}$ d) $\mathbb{D}$ e) $\mathbb{E}$





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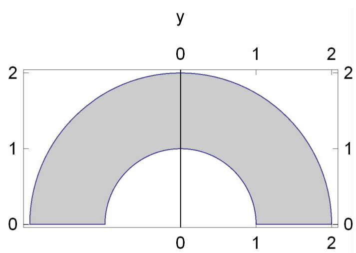
14 When $\mathbb{D}$ is the region shown below, the area-weighted integral $\int_{\mathbb{D}} \sqrt{x^2+y^2}\ da =$

a) $\frac{3\pi}{2}$ b) 4π c) π d) $\frac{7\pi}{3}$ e) $\frac{7\pi}{6}$

$r^2 = x^2 + y^2$



$$\int_1^2\int_0^\pi r\cdot r\,d\theta\,dr$$

$$= \int_1^2 r^2\int_0^\pi d\theta\,dr = \int_1^2 r^2(\pi)\,dr$$

$$= \pi\int_1^2 r^2\,dr = \pi\frac{1}{3}(r^3)\Big|_1^2$$

$$= \frac{\pi}{3}(8-1) = \frac{7\pi}{3}$$

$v = |\vec{V}| \leftarrow$ speed is magnitude of velocity

15 Which curve has constant speed and non-zero acceleration:

a) None, constant speed implies zero acceleration.

b) $x = \text{Sin}[t]$, $y = \text{Cos}[t]$, $z = \text{Cos}[t]$

c) $x = 3 + 2t$, $y = 2 + 3t$, $z = 2 - 5t$

d) $x = \text{Cos}[t]$, $y = 2t$, $z = \text{Sin}[t]$

e) $x = \text{Sin}[2t]$, $y = t$, $z = 2\,\text{Cos}[t]$

$$\frac{dv}{dt} = 0 \leftarrow \text{looking for this}$$

$$V = \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} \qquad A = \begin{pmatrix} -\sin t \\ -\cos t \\ -\cos t \end{pmatrix}$$

$$V \cdot A = -\sin\cancel{\cos} + \sin\cancel{\cos} + \sin\cos \neq 0$$

$$V = \begin{pmatrix} -\sin t \\ 2 \\ \cos t \end{pmatrix} \qquad A = \begin{pmatrix} -\cos t \\ 0 \\ -\sin t \end{pmatrix}$$

$$V \cdot A = \sin\cancel{\cos} + 0 - \sin\cancel{\cos} = 0 \checkmark$$

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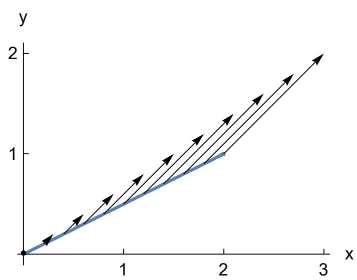
16 The work done by the force field $F\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x \\ 2y \end{pmatrix}$ **moving *along* the line segment from (0, 0) to (2, 1) is:**

a) 1 b) 2 c) $\frac{3}{2}$ d) 3 e) 4

(This is the same as flow along. HINT: This can be done two ways.)



$$P = \tfrac{1}{2}x^2 + y^2$$

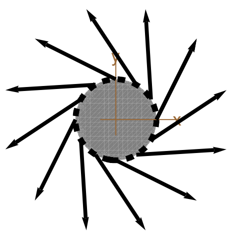
$$\nabla P = \begin{pmatrix} x \\ 2y \end{pmatrix}$$

$$P(2,1) - P(0,0) = \tfrac{1}{2}4 + 1 - 0 - 0 = 3$$

17 The flow of the vector field $F\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x - 2y \\ x + 3y \end{pmatrix}$ *out* of the unit circle centered at the origin is:

a) π b) 2π c) 3π d) 4π e) 5π

$$\oint_{\partial D} F \cdot dx = \iint_D \mathrm{Div}(F)\,da = \iint (1+3)\,da$$
$$= 4 \iint da$$
$$= 4\pi$$



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18 The circulation of the vector field $F\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x - 2y \\ x + 3y \end{pmatrix}$ *around* of the counterclockwise unit circle centered at the origin is:

swirl

a) π b) 2π c) 3π d) 4π e) 5π

$$\oint_{\partial D} F \cdot dx = \iint_D \mathrm{swirl}(F)\,da = \iint_D (1-(-2))\,da = 3\iint_D da = 3\pi$$

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19 Which vector field conserves energy for all $\{x, y\}$?

a) $F[x, y] = \begin{pmatrix} 0 \\ x \end{pmatrix}$ b) $F[x, y] = \begin{pmatrix} y \\ x \end{pmatrix}$ c) $F[x, y] = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix}$ d) $F[x, y] = \begin{pmatrix} -y \\ x \end{pmatrix}$ e) $F[x, y] = \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$

$\Downarrow$ $\mathrm{swirl}(F) = 1 \neq 0$

$\Downarrow$ $\mathrm{swirl}(F) = 0$

$\Downarrow$

$P = xy$

$\nabla P = \begin{pmatrix} y \\ x \end{pmatrix}$ ✓

20 Which vector field is not given by a potential for all $\{x, y\} \neq \{0, 0\}$**, that is, has** ***no*** **function** $p[x, y]$ **with** $\nabla p[x, y] = F[x, y]$

a) $F\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \frac{1}{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix}$ b) $F\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix} x \\ y \end{pmatrix}$ c) $F[x, y] = \sqrt{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix}$

d) $F[x, y] = \frac{1}{\sqrt{x^2+1}}\begin{pmatrix} x \\ 0 \end{pmatrix}$ e) $F[x, y] = \sqrt{x^2+y^2}\begin{pmatrix} 0 \\ x \end{pmatrix}$

(a) $\text{Swirl}(F) = \frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2} = 0$

(b) $\text{Swirl}(F) = 0$

(c) $\text{Swirl}(F) = 0$

(d) $\text{Swirl}(F) = 0$

(e) $\text{Swirl}(F) = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}} \neq 0$

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21 The function $f[x] = \frac{\text{Sin}[x]}{x}$ **is undefined at** $x = 0$ **by this formula, but if you write the series for sine and divide that by** x**, you obtain a series:** $f[x] = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$

$a_2 =$

a) 0 b) 1/2 c) -1/2 d) -1/3 e) -1/6

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$\downarrow$$

$$a_2 x^2 = -\frac{x^2}{3!}$$

$$a_2 = -\frac{1}{6}$$

22 Find the radius of convergence of the series $\mathrm{Log}\left[1+\frac{x}{3}\right]=\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\frac{x^4}{324}+\ldots=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{x}{3}\right)^k$

a) 0 b) 1 c) ∞ d) 3 e) $\frac{1}{3}$

$$r=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k-1}}{k}\left(\frac{1}{3}\right)^k}{\frac{(-1)^k}{k+1}\left(\frac{1}{3}\right)^{k+1}}\right|=\lim_{k\to\infty}\left|\frac{\frac{1}{k}\left(\frac{1}{3}\right)^k}{\frac{1}{k+1}\left(\frac{1}{3}\right)^{k+1}}\right|$$

$$=\lim_{k\to\infty}\frac{1}{(1/3)}\left|\frac{\frac{1}{k}\left(\frac{1}{3}\right)^k}{\frac{1}{k+1}\left(\frac{1}{3}\right)^k}\right|=\lim_{k\to\infty}3\left|\frac{k+1}{k}\right|=3$$

23 You are given

$$f[x]=\frac{2\,x+\mathrm{Log}[1+2\,x]+2\,x\,\mathrm{Log}[1+2\,x]}{2\,x\,(1+2\,x)}=2-3\,x+\frac{16}{3}\,x^2-10\,x^3+\frac{96}{5}\,x^4-\frac{112}{3}\,x^5+\frac{512}{7}\,x^6+\ldots=\sum_{k=0}^{\infty}(-2)^k\,\frac{k+2}{k+1}\,x^k$$

The third derivative at zero, $f'''[0]=$

a) $\frac{8}{9}$ b) $\frac{4}{5}$ c) $\frac{5}{3}$ d) $-\frac{5}{3}$ e) -60

$f'(0)=-3$

$f''(0)=\frac{16}{3}\cdot 2$

$f'''(0)=-10\cdot 3\cdot 2=-60$

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24 $e^{y^2}=c_0+c_1\,y+c_2\,y^2+c_3\,y^3+c_4\,y^4+c_5\,y^5+\cdots+c_n\,y^n+\cdots$

$c_4=$

a) 0 b) 1 c) $\frac{1}{2}$ d) $\frac{1}{6}$ e) $\frac{1}{24}$

$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots$

$e^{y^2}=1+y^2+\frac{y^4}{2}+\ldots$

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25 If $i=\sqrt{-1}$ **, the complex number, the exact numerical value of** $e^{i\pi}+\sum_{k=0}^{\infty}\frac{1}{2^k}=$

a) 0 b) $\frac{1}{2}$ c) 1 d) $i+1$ e) 2

Geometric Series

$e^{i\pi}=\cos(\pi)+i\sin(\pi)=-1+0=-1$

$\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{1}{1-\frac{1}{2}}=\frac{1}{1/2}=2$

$e^{i\pi}+\sum_{k=0}^{\infty}\frac{1}{2^k}=-1+2=1$