Midterm 2 Study Guide

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Below is a list of material to know for Midterm 2. Notes can be found under Files in the lecture's ICON module. For clarity of any topic, please refer to the book (found in MyLab). You can also email me.

(1) Section 3.1 - Limits
   (i) One-sided limits
   (ii) Limit of a function
   (iii) Existence of limits
   (iv) Piecewise functions
   (v) Rules for limits
   (vi) Limits at infinity and finding limits at infinity

(2) Section 3.2 - Continuity
   (i) Determine where a function is continuous
   (ii) Determine where a function is discontinuous
   (iii) Removable discontinuity
   (iv) Continuity on an open interval
   (v) Continuity on a closed interval
   (vi) Intermediate value theorem

(3) Section 3.3 - Rates of Change
   (i) Average rate of change
   (ii) Average speed
   (iii) Instantaneous rate of change
   (iv) Velocity

(4) Section 3.4 - Definition of the Derivative
   (i) calculate a tangent line
   (ii) calculate a secant line
   (iii) slope of curves
   (iv) differentiation using the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   (v) equivalent expressions for the definition of the derivative
   (vi) rationalizing the numerator
   (vii) existence of the derivative
   (viii) Left and right derivatives

(5) Section 3.5 - Graphical Differentiation

   (i) sketch the graph of the derivative given the graph of the original function

(6) Section 4.1 - Techniques for Finding Derivatives

   (i) notations for the derivative
   (ii) constant rule
   (iii) linearity rules for derivatives
      - derivative of a constant times a function
      - sum or difference rule

(7) Section 4.2 - Derivatives of products and quotients

   (i) product rule
   (ii) quotient rule

(8) Section 4.3 - The chain rule

   (i) composition of functions
   (ii) chain rule

(9) Section 4.4 - Derivatives of exponential functions

   (i) derivative of e^x
   (ii) derivative of a^x for any positive $a \neq 1$
   (iii) derivative of $a^{g(x)}$ and $e^{g(x)}$

(10) Section 4.5 - Derivatives of Logarithmic Functions

   (i) derivative of $\log_a x$, $\ln x$, $\log_a |g(x)|$, and $\ln |g(x)|$

(11) Section 4.6 - Derivatives of Trigonometric functions

   (i) derivative of $\sin x$, $\cos x$, $\tan x$
   (ii) know how to take the derivative of composition of functions with trigonometric functions
   (iii) double check to see if you may need to know the derivative of $\csc x$, $\sec x$, $\cot x$

(12) Section 5.1 - Increasing and Decreasing functions

   (i) critical numbers, critical points
   (ii) increasing and decreasing functions
   (iii) open intervals where a functions is increasing or decreasing

(13) Section 5.2 - Relative Extrema

   (i) relative (or local) maximum
   (ii) relative (or local) minimum
   (iii) relative (or local) extremum
   (iv) first derivative test

(14) Section 5.3 - Higher Derivatives, Concavity, and the Second Derivative Test

   (i) Second derivative (you should also know how to take the third and fourth derivative of a function)
   (ii) Suppose $s(t)$ describes the position of an object at time t. Then $s'(t)$ describes the velocity at time t and $s''(t)$ describes the acceleration at time t.
   (iii) concave upward, concave downward
   (iv) test for concavity

(v) a point where a graph changes concavity is called an inflection point
(vi) second derivative test

(15) Section 5.4 - Curve Sketching
   (i) be able to graph of a function noting any local extrema, plotting inflection points, making note of concavity, making note of any symmetry of a function
   (ii) applying first and second derivative tests will be necessary

(16) Section 6.1 - Absolute Extrema
   (i) absolute maximum
   (ii) absolute minimum
   (iii) absolute extremum
   (iv) extreme value theorem
   (v) critical point theorem

(17) Section 6.2 - Applications of Extrema
   (i) problems are relevant to previous section
   (ii) solving an applied extrema problem
   (iii) maximizing
   (iv) minimizing

(18) Section 6.3 - Implicit Differentiation
   (a) explicit function
   (b) Implicit function
   (c) Implicit differentiation
   (d) Tangent line

Practice problems are from sections 3.1 - 3.3

(1) Find $\lim_{x\to 2} g(x) = \frac{x^3-2x^2}{x-2}$.

(2) Determine $\lim_{x\to 2} h(x)$ where

$$h(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

(3) Find $\lim_{x\to -2}$ where

$$f(x) = \frac{3x+2}{2x+4}.$$

(4) Find $\lim_{x\to 0} \frac{|x|}{x}$.

(5) Suppose $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = 4$. Use the limit rules to find the following limits.

(a) $\lim_{x\to 2}[f(x) + 5g(x)]$.

(b) $\lim_{x\to 2} \frac{[f(x)]^2}{\ln g(x)}$.

(6) Use the properties of limits to help decide whether each limit exists. If a limit exists, find its value.

(a) $\lim_{x\to 3} \frac{x^3-9}{x-3}$

(b) $\lim_{x\to 0} \frac{1-\cos^2(x)}{\sin^2(x)}$ (Hint: consider trig identities.)

(c) $\lim_{x\to\infty} \frac{x^2+2x-5}{3x^2+2}$

(d) $\lim_{x\to\infty} \frac{2x^2-7x^4}{9x^2+5x-6}$

(7) Find all values of x where the function is discontinuous. For each value of x, give the limit of the function at that value of x. Be sure to note when the limit doesn't exist.

(a) $f(x) = \frac{5+x}{x(x-2)}$

(b) $f(x) = \frac{x^2-25}{x+5}$

(c) $p(x) = x^2 - 4x + 11$

(d) $f(x) = \sin(\frac{x}{x+2})$

(e) $j(x) = \ln|\frac{x+2}{x-3}|$

(f) $g(x) = \tan(\pi x)$

(8) Graph the given function, find all values of x where the function is discontinuous, and find the limit from the left and from the right at any values of x that are discontinuous.

(a)

$$f(x) = \begin{cases} 1 & \text{if } x < 2 \\ x+3 & \text{if } 2 \leq x \leq 4 \\ 7 & \text{if } x > 4 \end{cases}$$

(b)

$$h(x) = \begin{cases} x^2 + x - 12 & \text{if } x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

(9) Find the value of the constant k that makes the function continuous.

(a)

$$g(x) = \begin{cases} \frac{3x^2+2x-8}{x+2} & \text{if } x \neq -2 \\ 3x + k & \text{if } x = -2 \end{cases}$$

(b)

$$f(x) = \begin{cases} kx^2 & \text{if } x \le 2 \\ x + k & \text{if } x > 2 \end{cases}$$

(10) Find the average rate of change for each function over the given interval.

(a) $y = -4x^2 - 6$ between $x = 1$ and $x = 3$

(b) $y = \frac{1}{x-1}$

(c) $y = \sqrt{x}$ between $x = 1$ and $x = 4$

(d) $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

(11) Find the instantaneous rate of change for each function at the given value.

(a) $f(x) = x^2 + 2x$ at $x = 0$

(b) $s(t) = -4t^2 - 6$ at $t = 2$

(c) $g(t) = 1 - t^2$ at $t = -1$

(d) $F(x) = x^3 + 2$ at $x = 0$

(12) If the instantaneous rate of change of $f(x)$ with respect to x is positive when $x = 1$, is f increasing or decreasing there?

(13) Suppose the position of an object moving in a straight line is given by $s(t) = t^3 + 2t + 9$. Find the instantaneous velocity at each time.

(a) t=1

(b) t=4

Problems are from sections 3.4, 3.5, and 4.1

(1) If $f(x) = \frac{x^2-1}{x+2}$, where is f not differentiable?

(2) Using the definition of the derivative, find $f'(x)$. Then find $f'(-2)$, $f'(0)$, and $f'(3)$ when the derivative exists.

(a) $3x - 7$

(b) $f(x) = -4x^2 + 9x + 2$

(c) $f(x) = 12/x$

(d) $f(x) = -3\sqrt{x}$

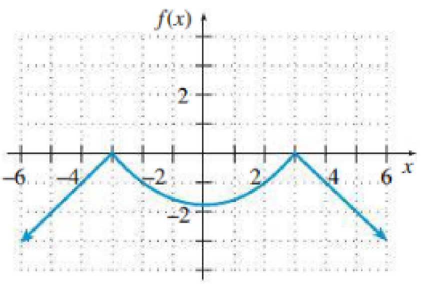
(e) $f(x) = 4x^3 - 3$

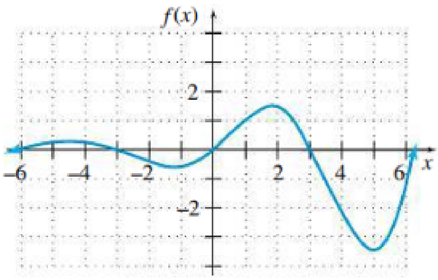
(3) For each function, find (a) the equation of the secant line through the points where x has the given values, and (b) the equation of the tangent line when x has the first value.

(a) $f(x) = x^2 + 2x$; $x = 3$, $x + 5$

(b) $f(x) = \frac{5}{x}$; $x = 2$, $x = 5$

(4) Sketch the graph of the derivative for each function shown.





(5) Find the derivative of each function defined as follows. (You may use any rule covered in lecture or discussion.)

(a) $y = 12x^3 - 8x^2 + 7x + 5$
(b) $y = -100\sqrt{x} - 11x^{2/3}$
(c) $y = \frac{3}{x^6} + \frac{1}{x^5} - \frac{7}{x^2}$
(d) $f(x) = \frac{x^3+5}{x}$

(6) For each function find all values of x where the tangent line is horizontal. (You may use any rule covered in lecture or discussion.)

(a) $f(x) = 2x^3 + 9x^2 - 60x + 4$
(b) $f(x) = x^3 - 5x^2 + 6x + 3$

Practice problems are from sections 4.2, 4.3, and 4.4

(1) Use the product rule to find the derivative of the following.

(a) $g(t) = (3t^2 + 2)^2$
(b) $y = (2x - 3)(\sqrt{x} + 2)$
(c) $q(x) = (x^{-2} - x^{-3})(3x^{-1} + 4x^{-4})$

(2) Use the quotient rule to find the derivative of the following.

(a) $f(x) = \frac{6x+1}{3x+10}$
(b) $y = \frac{9-7t}{1-t}$
(c) $f(x) = \frac{(3x^2+1)(2x-1)}{5x+4}$

(3) When a certain drug is injected into a muscle, the muscle responds by contracting. The amount of contraction, s (in millimeters), is related to the concentration of the drug, x (in millimeters), by

$$s(x) = \frac{x}{m + nx},$$

where m and n are constants.

(a) Find $s'(x)$.
(b) Find the rate of contraction when the concentration of the drug is 50 ml, $m = 10$, and $n = 3$.

(4) *Composition:* Let $f(x) = 5x^2 - 2x$ and $g(x) = 8x + 3$. Find the following.

(a) $f[g(2)]$
(b) $g[f(-5)]$
(c) $f[g(k)]$
(d) $g[f(5z)]$

(5) Find the derivative of each function defined as follows.

(a) $y = (8x^4 - 5x^2 + 1)^4$

(b) $f(x) = -7(3x^4 + 2)^{-4}$

(c) $f(t) = 8\sqrt{4t^2 + 7}$

(d) $r(t) = \frac{(5t-6)^4}{3t^2+4}$

(6) Find the equation of the tangent line to the given function at the given value of x.

(a) $f(x) = \sqrt{x^2 + 16}; \quad x = 3$

(b) $f(x) = x(x^2 - 4x + 5)^4; \quad x = 2$

(7) Find the derivatives of the functions defined as follows.

(a) $y = (x + 3)^2 e^{4x}$

(b) $p = \frac{500}{12+5e^{-0.5t}}$

(c) $s = 2 \cdot e^{\sqrt{t}}$

(d) $y = \frac{te^t+2}{e^{2t}+1}$

(e) $f(x) = exp(x^2/(x^3 + 2))$; (Here, the notation $exp(x) := e^x$.)

(8) Prove that if $y = y_0 e^{kt}$, where y_0 and k are constants, then $dy/dt = ky$. (*Interpretation of result:* This says that for exponential growth and decay, the rate of change of the population, is proportional to the size of the population, and the constant of proportionality is the growth or decay constant.)

(9) It has been observed that the following formula accurately models the relationship between the size of a breast tumor and the amount of time that is has been growing.

$$V(t) = 1100[1023e^{-0.02415t} + 1]^{-4},$$

where t is in months and $V(t)$ is measured in cubic centimeters.

(a) Find the tumor volume at 240 months.

(b) Assuming that the shape of a tumor is spherical, find the radius of the tumor from part a. (*Hint:* The volume of a sphere is given by the formula $V = (3/4)\pi r^3$.)

(c) If a tumor of size 0.5 cm^3 is detected, according to the formula, how long has it been growing? What does this imply?

(d) Find $\lim_{t\to\infty} V(t)$ and interpret this value. Explain whether this makes sense.

(e) Calculate the rate of change of tumor volume at 240 months and interpret.

(10) In a model of the nervous system, the intensity of excitation, I, of a nerve pathway is given by

$$I = E[1 - exp(-a(S - h)/E)]$$

where E is the maximum possible excitation, S is the intensity of a stimulus, h is a threshold stimulus, and a is a constant. Find the rate of change of the intensity of the excitation with respect to the intensity of the stimulus. (Here, the notation $exp(x) := e^x$.)

Practice problems from sections 4.5, 4.6, and 5.1

(1) Find the derivative of the following functions.

(a) $y = \ln(8 - 3x)$

(b) $y = \ln\sqrt{2x + 1}$

(c) $y = \dfrac{\ln x}{4x + 7}$

(d) $y = \log(6x)$

(e) $f(x) = e^{\sqrt{x}} \ln(\sqrt{x} + 5)$

(f) $f(t) = \dfrac{\ln(t^2+t)+t}{\ln(t^2+1)}$

(g) $y = 12\tan(9x+1)$

(h) $y = \cos^4 x$

(i) $y = \dfrac{\tan x}{x-1}$

(j) $y = \sin(\ln 3x^4)$

(k) $y = 3\tan(\frac{1}{4}x) + 4\sin(2x) - 5\cos(x) + e^{-2x}$

(2) Verify that $(\tan x)' = \sec^2 x$ by using the quotient rule and the fact that $\tan x = \dfrac{\sin x}{\cos x}$. (*Hint:* $\sin^2 x + \cos^2 x = 1$.)

(3) For each function, find (i) the critical numbers; (ii) the open intervals where the function is increasing; and (iii) the open intervals where the function is decreasing.

(a) $y = 2.3 + 3.4x - 1.2x^2$

(b) $f(x) = \frac{2}{3}x^3 - x^2 - 24x - 4$

(c) $f(x) = 4x^3 - 15x^2 - 72x + 5$

(d) $y = 6x - 9$

(e) $y = -3x + 6$

(f) $y = x^{2/3} - x^{5/3}$

(g) $y = \sin x$

(h) $y = 3\sec x$

Practice problems are from sections 5.2, 5.3, and 5.4

(1) Find the x-value of all points where the functions defined as follows have any relative extrema. Find the value(s) of any relative extrema.

(a) $f(x) = x^2 - 10x + 33$

(b) $f(x) = x^2 + 8x + 5$

(c) $f(x) = -\frac{4}{3}x^3 - \frac{21}{2}x^2 - 5x + 8$

(d) $f(x) = x^4 - 18x^2 - 4$

(e) $2x + \ln x$

(f) $f(x) = \dfrac{2^x}{x}$

(g) $f(x) = sin(\pi x)$

(2) Find the second derivative of the following functions.

(a) $f(x) = 3x^2 - 4x + 8$

(b) $f(x) = 32x^{3/4}$

(c) $f(x) = 5e^{-x^2}$

(d) $f(x) = \dfrac{\ln x}{4x}$

(3) Let f be an nth-degree polynomial of the from

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

(a) Find $f^{(n)}(x)$.

(b) Find $f^{(k)}(x)$ for $k > n$.

(4) Find the open interval where the functions are concave upward or concave downward. Find any inflection points. Make note of any symmetry (function is either even, odd, or neither). Make note of any asymptotes (vertical and horizontal), if there are any.

(a) $f(x) = x^2 + 10x - 9$
(b) $f(x) = 8 - 6x - x^2$
(c) $f(x) = -2x^3 + 9x^2 + 168x - 3$
(d) $f(x) = -x^3 - 12x^2 - 45x + 2$
(e) $f(x) = \dfrac{3}{x-5}$
(f) $18x - 18e^{-x}$
(g) $f(x) = x^2 \log|x|$
(h) $f(x) = 2x$
(i)
(j) $f(x) = \sin(2x)$

(5) Sketch the following functions from the previous problem. Ensure your sketch shows critical points, inflection points, intervals where $f(x)$ is increasing and decreasing,and intervals where $f(x)$ is concave up and concave down. Point out any local extrema.

(a) $f(x) = x^2 + 10x - 9$
(b) $f(x) = 8 - 6x - x^2$
(c) $f(x) = -2x^3 + 9x^2 + 168x - 3$
(d) $f(x) = -x^3 - 12x^2 - 45x + 2$

Practice problems are from sections 6.1 and 6.2

(1) Find the absolute extrema if they exist, as well as all values of x where they occur.

(a) $f(x) = 2x + \frac{8}{x^2} + 1, x > 0$
(b) $f(x) = 12 - x - \frac{9}{x}, x > 0$
(c) $f(x) = -3x^4 + 8x^3 + 18x^2 + 2$
(d) $f(x) = \dfrac{x-1}{x^2 + 2x + 6}$
(e) $f(x) = \dfrac{\ln x}{x^3}$

(2) A piece of wire 12 feet long is cut into two pieces – one of length x and the other of length $12 - x$.

|—-x—-| + |———$12 - x$———| = |————–12————-|

One piece is made into a circle and the other piece is made into a square. Let the piece of length x be formed into a circle. We allow x to equal 0 or 12, so all the wire may be used for the square or for the circle.

The radius of the circle is given by $\frac{x}{2\pi}$.

The area of the circle is given by $\pi(\frac{x}{2\pi})^2$.

The side of the square is given by $\frac{12-x}{4}$.

The area of the square is given by $(\frac{12-x}{4})^2$.

(a) Where should the cut be made in order to minimize the sum of the areas enclosed by both figures?
(b) Where should the cut be made in order to make the sum of the areas maximum? (*Hint:* Remember to use the endpoints of a domain when looking for absolute maxima and minima.)
(c) From the solution in part (a), show that the side of the square equals the diameter of the circle, that is, that the circle can be inscribed in the square.

(3) Using the steps below, find non-negative numbers x and y that satisfy the given requirements. Give the optimum value of the indicated expression. $x + y = 180$ and the product $P = xy$ is as large as possible.

(a) Solve $x + y = 180$ for y.

(b) Substitute the result from part (a) into $P = xy$, the equation for the variable that is to be maximized.

(c) Find the domain of the function P found in part (b).

(d) Find dP/dx. Solve the equation $dP/dx = 0$.

(e) Evaluate P at any solutions found in part (d), as well as at the endpoints of the domain found in part (c).

(f) Give the maximum value of P, as well as the two numbers x and y whose product is that value.

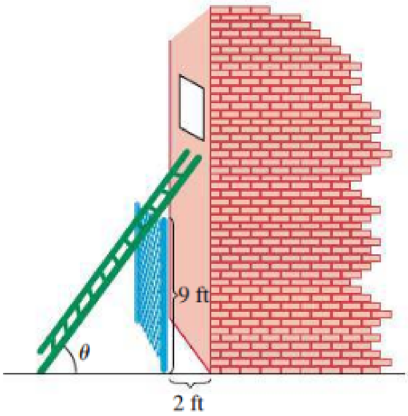
(4) Use the steps shown in the previous problem to find non-negative numbers x and y that satisfy the given requirements. Give the optimum value of the indicated expression.

(a) The sum of x and y is 140 and the sum of the squares of x and y is minimized.

(b) $x + y = 90$ and x^2y is maximized.

(c) $x + y = 105$ and xy^2 is maximized.

(5) A thief tries to enter a building by placing a ladder over 9-ft-high fence so it rests against the building, which is 2 ft back from the fence. (See the figure below.) What is the length of the shortest ladder that can be used? (*Hint:* Let θ be the angle between the ladder and the ground. Express the length of the ladder in terms of θ, and then find the value of θ that maximizes the length of the ladder.)



Questions from Section 6.3.

(1) Find dy/dx by implicit differentiation for the following.

(a) $6x^2 + 5y^2 = 36$

(b) $8x^2 - 10xy + 3y^2 = 26$

(c) $2\sqrt{x} + 4\sqrt{y} = 5y$

(d) $e^{x^2y} = 5x + 4y + 2$

(e) $x + \ln y = x^2y^3$

(f) $\sin(xy) = x$

(2) Find the equation of the tangent line at the given point on each curve.

(a) $x^2 + y^2 = 25; \quad (-3, 4)$

(b) $x^2y^2 = 1; \quad (-1, 1)$

(c) $y + \dfrac{\sqrt{x}}{y} = 3; \quad (4, 2)$